Lattice model in three dimensions with a θ term

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We study a three-dimensional Abelian lattice model in which the analogue of a theta term can be defined. This term is defined by introducing a neutral scalar field, and its effect is to couple magnetic monopoles to the scalar field and vortices to the gauge field. An interesting feature of this model is the presence of an exact duality symmetry that acts on a three-parameter space. It is shown that this model has an interesting phase structure at nonzero values of θ. In addition to the usual confinement and vortex phases there are phases in which loops with composite charges condense. The presence of novel pointlike excitations also alters the physical properties of the system.

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I. INTRODUCTION

Topology has played an important role in statistical mechanics and field theory. The topology of the dynamical variables leads to novel excitations that can have a profound effect in determining the physical properties of the system. A well-known example in statistical mechanics is the vortex excitation [1] in the two-dimensional planar model. These vortex excitations, which exist because of the angular nature of the spin variables, drive a phase transition which is very different from other phase transitions in statistical mechanics. In gauge theories, the analogue of the vortex excitation is the magnetic monopole [2]. The magnetic monopole exists as a consequence of the topology of the gauge group. Multimonopole states are also present as collective excitations. These multimonopole excitations can form a plasma phase in which electric charges are confined. A classic example of this phenomenon is compact QED in three Euclidean dimensions [3]. Apart from the intrinsic topology of the dynamical variables, we can also consider terms which have a direct topological significance. One such term is the well-known θ term in non-Abelian gauge theories. For the SU(2) non-Abelian gauge theory, this term is defined as

$$\frac{\theta}{32\pi^2}\int d^4x \mathrm{tr} F_{\mu\nu}\tilde{F}_{\mu\nu}. \tag{1}$$

Although this term is a total derivative, it can have a nontrivial effect whenever long range fields are present. In the space of finite action configurations, the above term is precisely the winding number of mappings from S_3 to S_3. Gauge field configurations which give a nonzero value to the expression in Eq. (1) can affect some of the physical properties of the system [4]. An important effect of the term in Eq. (1) is that it can convert a magnetic monopole into a particle with an additional electric charge—a dyon [5]. It was conjectured in [6] that non-Abelian gauge theories could have new phases of oblique confinement as a result of the interactions between these dyons. The analogue of the term in Eq. (1) in an Abelian gauge theory is

$$\int d^4x F_{\mu\nu}\tilde{F}_{\mu\nu}. \tag{2}$$

Unlike in the non-Abelian theory, this term does not have topological significance as the winding number of any mapping. However, this term can have a nontrivial physical effect in the presence of magnetic monopoles. The effect of the θ term in Eq. (2) was studied on the lattice in [7]. It was shown that the θ term drastically alters the phase structure of the theory and a rich phase structure was uncovered as a function of θ. The oblique confinement phases conjectured in [6] were also elucidated. An exact duality symmetry [which is the action of the group SL(2,Z)] was demonstrated to hold in this model [8], and the action of this symmetry was used to predict the entire phase structure of the model. Both the θ terms discussed so far require four Euclidean dimensions. In [7,8] some two-dimensional spin models were also considered in which a θ term could be defined. We would like to consider such a term in three Euclidean dimensions and study its effects. An inspection of the properties of the ϵ tensor shows that defining gauge invariant terms using only pure gauge fields is not possible. An exception is the Cherns-Simons term, which can be defined using only the ϵ term and gauge fields [9]. Lattice models for the Chern-Simons term have been considered in [10]. The θ term that we will consider in three dimensions is the lattice analogue of the following term:

$$i\theta\int d^3x \epsilon_{\mu\nu\lambda}F_{\mu\nu}\partial_\lambda\phi. \tag{3}$$

It is clear that, under the parity transformation in three dimensions,

$$x\rightarrow x,$$

$$y\rightarrow -y,$$

$$t\rightarrow t,$$

the θ term changes sign, i.e., $\theta\rightarrow-\theta$. Therefore, the θ term violates parity unless the free energy is an even function of θ. The field ϕ is a neutral field and hence the above term is gauge invariant. A simple integration by parts gives the following terms:

$$\partial_\lambda[\epsilon_{\mu\nu\lambda}F_{\mu\nu}(x)\phi(x)]-[\epsilon_{\mu\nu\lambda}\partial_\lambda F_{\mu\nu}(x)]\phi(x). \tag{4}$$

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The dual of the field strength is defined as

$$\tilde{F}_\lambda(x) = \tfrac{1}{2}\epsilon_{\lambda\mu\nu}F_{\mu\nu}. \tag{5}$$

The second term is seen to be $2\partial_\lambda \tilde{F}_\lambda(x)\phi(x)$ and this term can be nonzero in the presence of magnetic monopoles. In the presence of magnetic monopoles,

$$\partial_\lambda \tilde{F}_\lambda = \tilde{m}(x), \tag{6}$$

where $\tilde{m}(x)$ is the magnetic monopole density at x. The second term leads to the coupling

$$2\theta i \tilde{m}(x)\phi(x). \tag{7}$$

Another nontrivial contribution to Eq. (3) can come from a vortex line. This is seen by doing the integration by parts in Eq. (3) differently as

$$2\epsilon_{\mu\nu\lambda}\partial_\mu(A_\nu\partial_\lambda\phi) - 2\epsilon_{\mu\nu\lambda}A_\nu\partial_\mu\partial_\lambda\phi. \tag{8}$$

The second term can be written as

$$-2\theta i \int m_\mu(x)A_\mu(x). \tag{9}$$

The quantity

$$m_\mu(x) = \epsilon_{\mu\lambda\nu}\partial_\lambda\partial_\nu\phi \tag{10}$$

is nonzero around a vortex line. Hence, the term in Eq. (3) is nonzero in the presence of magnetic monopoles and vortex lines, and introduces new couplings between the topological $[\tilde{m}_\mu(\star x)$ and $\tilde{m}(\star x)]$ and the spin-wave (A_μ and ϕ) degrees of freedom. In the absence of these topological excitations, the term in Eq. (3) will have no physical effect. Another way of motivating the term in Eq. (3) is by dimensionally reducing the four-dimensional θ term in Eq. (2). At high temperatures, the leading order contribution from a term like Eq. (2) is given by Eq. (3). In this paper, we will present an analysis of the θ term in three Euclidean dimensions. In three dimensions, the physical properties of this model are quite different from the four-dimensional one. We will show that many of the interesting features pointed out in [7,8] are also present in three dimensions. However, there are also significant differences. As already explained before, the θ term becomes important whenever there are magnetic monopoles or vortex lines. On the lattice we can naturally define models which contain monopoles and vortex lines as possible excitations. This is possible if the degrees of freedom are considered as angular variables. We first present two well-known lattice models which have monopoles and vortex excitations, and then we study the effect of the θ parameter in these models. We will see that the θ parameter couples these two models in a nontrivial way and leads to a rich phase structure. The analysis presented here is based on the technique of duality transformations as applied to statistical systems. These techniques have been used very effectively to understand systems like the planar model [1] and the U(1) lattice gauge theory [11]. Recently, in [12], dual transformations have also been constructed for non-Abelian lattice gauge theories.

The main aim of this paper is to present an analysis of the θ term in a lattice model containing both monopoles and vortex lines. The θ term is introduced on the lattice by coupling the monopoles to the scalar field and the vortex lines to the gauge field. Many of the techniques used for studying the lattice model are well known in the literature but we present many details for the sake of completeness. Also, at some places we have managed to give more illuminating derivations of some of the steps in the analysis. The organization of this paper is as follows. In Secs. I and II we discuss lattice models which have monopoles and vortices as possible excitations. In Sec. III we present a detailed analysis of the model obtained by adding a θ term. In Sec. IV we make some concluding remarks. Some technical details are presented in the Appendix.

II. LATTICE MODEL WITH MONOPOLES

The model considered here is given by the following action:

$$S_1 = \frac{-\beta_g}{2}\sum_{x\mu\nu}[\partial_\mu\phi_\nu - \partial_\nu\phi_\mu - 2\pi s_{\mu\nu}(x)]^2 + ip\sum_{x\mu} m_\mu(x)\phi_\mu(x). \tag{11}$$

The partition function is given by

$$Z_1 = \mathrm{tr}\exp(S_1). \tag{12}$$

The symbol ‘‘tr’’ denotes

$$\sum_{m_\mu=-\infty}^{\infty}\sum_{s_{\mu\nu}=-\infty}^{\infty}\int_{-\infty}^{\infty} d\phi_\mu. \tag{13}$$

The variables ϕ_μ and m_μ are defined on the links of the lattice and the integer-valued variables $s_{\mu\nu}$ are defined on the plaquettes of the lattice. The symbol x denotes a three-dimensional vector and the symbol ∂_μ is the lattice derivative. The fields ϕ_μ are the gauge degrees of freedom whereas the integer valued variables $s_{\mu\nu}$ are the monopole degrees of freedom. The above model describes gauge fields coupled minimally to current loops with the additional presence of magnetic monopoles. This model has the following gauge invariance:

$$\phi_\mu(x) \to \phi_\mu + \partial_\mu\Lambda. \tag{14}$$

This also requires that

$$\partial_\mu m_\mu = 0. \tag{15}$$

Hence, the summation over m_μ is restricted to a summation over closed loops. To see that this model describes magnetic monopoles, consider the quantity

$$F_{\mu\nu}(x) = \partial_\mu\phi_\nu(x) - \partial_\nu\phi_\mu(x) - 2\pi s_{\mu\nu}(x). \tag{16}$$

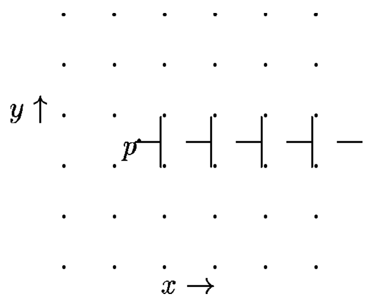

FIG. 1. The string attached to the dual excitations runs on the dual lattice.

Consider a configuration in which $s_{\mu\nu}(x)\neq 0$ on all the plaquettes pierced by the string in Fig. 1. This figure shows the projection of the string on the XY plane (as dashed lines) and the projection of the plaquettes (which are in the YZ plane) on the XY plane (as solid lines). The string is on the dual lattice and begins at P and extends to infinity in the z direction. It is easily seen that, in this configuration,

$$\partial_\lambda \tilde{F}_\lambda = -2\pi\delta(x_P), \tag{17}$$

this means that there is a magnetic monopole at P with a Dirac string in the z direction. Thus, the integer-valued plaquette degrees of freedom account for magnetic monopoles. The monopole and its associated Dirac string reside on the dual lattice. The monopoles in this model can be made explicit by means of a duality transformation. A duality transformation generally involves three steps. First, the quadratic part of the action is linearized by introducing an auxiliary field. The next step is to integrate over the original (in this case ϕ_μ) degrees of freedom and this will lead to a constraint for ϕ_μ. This constraint is solved by introducing degrees of freedom which are the dual variables and this solution is inserted back into the partition function. A few manipulations after that lead to the following form of the partition function:

$$\begin{aligned} Z_1 = \mathrm{Tr}\exp\left(-8\pi^2\beta_g\sum_{x,x'}\tilde{m}(x)G(x-x')\tilde{m}(x')\right) \\ \times\exp\left(\frac{-p^2}{8\beta_g}\sum_{x,x'}m_\mu(x)G(x-x')m_\mu(x')\right) \\ \times\exp\left(2\pi i\sum_{x,x'}\tilde{m}(x)G(x-x')\epsilon_{\mu\nu\lambda}\partial_\lambda K^\star_{\mu\nu}\right). \end{aligned} \tag{18}$$

The monopole density $\tilde{m}(\star x)$ is defined as

$$\tilde{m}(\star x) = \frac{1}{2}\epsilon_{\lambda\mu\nu}\partial_\lambda s_{\mu\nu}(x). \tag{19}$$

The symbol $\star x$ denotes that the monopole is defined on the dual lattice point. This definition requires that the total monopole number in the system be zero. The symbol ''Tr'' denotes

$$\sum_{m_\mu,\tilde{m}=-\infty,\infty}, \tag{20}$$

with the understanding that the summation is over configurations with zero total monopole number and closed loops of currents. The quantity $K^\star_{\mu\nu}$ is a particular solution of

$$\partial_\mu K^\star_{\mu\nu} = p m_\nu. \tag{21}$$

In the above expression for the partition function, $G(x-x')$ is the three-dimensional Green's function which satisfies

$$-\partial^2 G(x-x') = \delta(x-x'). \tag{22}$$

The symbol ∂^2 is the lattice Laplacian. The partition function in Eq. (18) describes a gas of current loops $[m_\mu(x)]$ and magnetic monopoles $[\tilde{m}(\star x)]$. The monopoles and current loops interact among themselves with a three-dimensional Coulomb interaction. The last term in Eq. (18) describes the interaction of a monopole with a current loop. This interaction is just the solid angle subtended by the current loop at the monopole and this is shown in the appendix. Before proceeding we briefly outline the steps leading to Eq. (18). The linearization of the action is accomplished by introducing an auxiliary field $K_{\mu\nu}$ and the partition function becomes

$$\begin{aligned} \int_{-\infty}^{\infty} dK_{\mu\nu}\mathrm{tr}\exp\left(i\sum_{x\mu\nu}K_{\mu\nu}(x)\right)[\partial_\mu\phi_\nu-\partial_\nu\phi_\mu-2\pi s_{\mu\nu}(x)] \\ \times\exp\left(\frac{-1}{2\beta_g}\sum_x K^2_{\mu\nu}(x)\right)\exp\left(ip\sum_{x\mu\nu}m_\mu(x)\phi_\mu(x)\right). \end{aligned} \tag{23}$$

Integration over ϕ_μ results in the constraint

$$2\partial_\mu K_{\mu\nu} = p m_\nu. \tag{24}$$

The solution of this constraint is

$$2K_{\mu\nu} = \epsilon_{\mu\nu\lambda}\partial_\lambda\phi + K^\star_{\mu\nu}. \tag{25}$$

Since $2K_{\mu\nu}$ has to be an integer, the fields $\phi(x)$ and $K^\star_{\mu\nu}(x)$ are integer valued. However, the solution to the constraint is not unique because

$$\phi(x)\rightarrow\phi(x)+c \tag{26}$$

also solves the constraint. Using this ''dual'' gauge invariance, the range of integration over ϕ can be taken from $-\infty$ to ∞. Substituting the solution of the constraint in Eq. (25) in Eq. (23) and then doing a Gaussian integration over ϕ leads to Eq. (18).

III. LATTICE MODEL WITH VORTICES

A lattice model describing vortices is defined by the action

$$S_2 = -\frac{\beta_h}{2}\sum_{x\mu}\,[\partial_\mu\theta(x) - 2\pi l_\mu(x)]^2 + ip\sum_r\, m(r)\theta(r). \tag{27}$$

The partition function is given by

$$Z_2 = \mathrm{tr}\exp(S_2). \tag{28}$$

The symbol ''tr'' denotes

$$\sum_{m,l_\mu=-\infty}^{\infty}\int_{-\infty}^{\infty} d\theta. \tag{29}$$

In the above model the variables θ and m are defined on the sites of the lattice and the variables n_μ are defined on the links of the lattice. θ's are the spin-wave degrees of freedom and n_μ's are the vortex degrees of freedom. The above model describes vortex lines interacting with charged $m(x)$ variables. The model has the following global invariance:

$$\theta(x) \to \theta(x) + c, \tag{30}$$

where c is any constant. This automatically requires

$$\sum_x\, m(x) = 0. \tag{31}$$

The vortex lines in this model can be identified by considering the quantity

$$V_\mu(x) = \partial_\mu\theta(x) - 2\pi l_\mu(x). \tag{32}$$

Consider a configuration in which l_μ is nonzero on all the links pierced by the world line of the string. This configuration is also shown in Fig. 1. This configuration is a vortex running in the z direction which is the direction out of the plane of the figure. The accompanying string is chosen in the X direction and is indicated by the dashed line which pierces the bonds between pairs of nearest neighbor sites (shown as solid lines). Any closed loop about this vortex line will give a nonzero value for

$$\sum_x\, V_\mu(x). \tag{33}$$

The vortices in this model can again be made explicit by means of a dual transformation. Introducing an auxiliary field as was done in the previous section and repeating the procedure described before, we get an expression for the partition function:

$$Z_2 = \mathrm{Tr}\exp\left(-\frac{p^2}{2\beta_h}\sum_{x,x'} m(x)G(x-x')m(x')\right) \times\exp\left(-2\pi^2\beta_h\sum_{x,x'}\tilde{m}_\mu(x)G(x-x')\tilde{m}_\mu(x')\right) \times\exp\left(-2\pi i\sum_{x,x'}\tilde{m}_\lambda(x)G(x-x')\epsilon_{\lambda\mu\nu}\partial_\mu K^\star_\nu\right). \tag{34}$$

The ''Tr'' denotes

$$\sum_{m,\tilde{m}=-\infty}^{\infty}. \tag{35}$$

The vorticity $\tilde{m}_\mu(\star x)$ is defined as

$$\tilde{m}_\mu(\star x) = \epsilon_{\mu\nu\lambda}\partial_\nu\tilde{\phi}_\lambda(x). \tag{36}$$

From the above equation it is clear that the vortices form closed loops because

$$\partial_\mu\tilde{m}_\mu(\star x) = 0. \tag{37}$$

As in the case of the monopole, the vortex lines and their associated strings reside on the dual lattice. A closed vortex loop in the $\mu\nu$ plane will have a sheet swept by its string and the plaquettes in this sheet will be dual to the links with $l_\mu \neq 0$. The steps leading to the dual transformation are analogous to those in the previous section; only the auxiliary field K_μ is now introduced on every link. The constraint equation that has to be solved is

$$\partial_\mu K_\mu(x) = pm(x). \tag{38}$$

The solution of this constraint equation is

$$K_\mu(x) = \epsilon_{\mu\nu\lambda}\partial_\nu\tilde{\phi}_\lambda + K^\star_\mu(x), \tag{39}$$

with $K^\star_\mu(x)$ being a solution of the inhomogeneous equation. The dual gauge invariance in this case is

$$\tilde{\phi}(x) \to \tilde{\phi}(x) + \partial_\lambda\Lambda. \tag{40}$$

The first two terms in Eq. (34) describe the Coulomb interaction between $m(x)$ variables and the vortex currents $\tilde{m}_\mu(\star x)$. [$G(x-x')$ is the same Green's function as in Eq. (22)]. The last term represents an interaction between the vortex currents and the m charges. $K^\star_\mu(x)$ is the solution of the inhomogeneous equation

$$\partial_\mu K^\star_\mu(x) = pm(x). \tag{41}$$

This interaction is again proportional to the solid angle (apart from a negative sign) of the vortex current subtended at m. A demonstration of this can be found in the Appendix.

Before we proceed to the model with a θ term, we can already see that the two models described above have a very similar structure. For instance, the monopole model has

$$\partial_\mu m_\mu = 0 \tag{42}$$

as the current conservation equation and

$$\sum_{x^\star} \tilde{m}(\star x) = 0 \tag{43}$$

as the monopole conservation equation. In the vortex model, the m charge conservation equation is

$$\sum_x m(x) = 0, \tag{44}$$

and the vortex conservation equation is

$$\partial_\mu \tilde{m}_\mu(\star x). \tag{45}$$

The roles of the conservation of ordinary charges and topological charges are clearly reversed and the two models are dual to each other. This duality will be made more precise in the next section.

IV. COUPLED MODEL

Now we can couple the two previous models by introducing a θ term as explained in the introduction. The θ term is defined by introducing two additional couplings as

$$S_\theta = \frac{ip\theta}{2\pi} \sum_{x,\mu} \tilde{m}(\star x)\theta(x) + \frac{ip\theta}{2\pi} \sum_{x,\mu} \tilde{m}_\mu(\star x)\phi_\mu(x). \tag{46}$$

The action of the coupled model is given by

$$S = S_1 + S_2 + S_\theta, \tag{47}$$

and can be written out as

$$\begin{aligned} S = & \frac{-\beta_g}{2} \sum_{x\mu\nu} [\partial_\mu \phi_\nu - \partial_\nu \phi_\mu - 2\pi s_{\mu\nu}(x)]^2 \\ & - \frac{\beta_h}{2} \sum_{x\mu} [\partial_\mu \theta(x) - 2\pi l_\mu(x)]^2 + ip \sum_{x\mu} m_\mu(x)\phi_\nu(r) \\ & + ip \sum_x m(x)\theta(x) + \frac{ip\theta}{2\pi} \sum_{x,\mu} \tilde{m}(\star x)\phi(x) \\ & + \frac{ip\theta}{2\pi} \sum_{x,\mu} \tilde{m}_\mu(\star x)\phi_\mu(x). \end{aligned} \tag{48}$$

The partition function of this model is given by

$$Z = \mathrm{tr}\exp(-S). \tag{49}$$

The trace represents the sum over states:

$$\sum_{m(x)-\infty}^{\infty} \sum_{m_\mu(x)-\infty}^{\infty} \int_{-\infty}^{\infty} d\phi_\mu(x) \int_{-\infty}^{\infty} d\theta(x). \tag{50}$$

When $\theta = 0$, the two models are decoupled and the partition function is simply a product of their separate partition functions:

$$Z = Z_1 Z_2. \tag{51}$$

The model at $\theta = 0$ represents a system of monopoles, currents, m charges, and vortices. However, the excitations in one system do not interact with those in the other system. The model at $\theta = 0$ can be shown to be trivially self-dual. This follows by noting that the transformations

$$\begin{aligned} \beta_g &\to \frac{p^2}{16\pi^2 \beta_h}, \\ \beta_h &\to \frac{p^2}{16\pi^2 \beta_g}, \\ m(x) &\to \tilde{m}(\star x), \\ m_\mu(x) &\to \tilde{m}_\mu(\star x) \end{aligned} \tag{52}$$

simply interchanges, the two expressions in Eq. (18) and Eq. (34). Note that this holds before the sum over states is performed in Eq. (18) and Eq. (34). The dual transformation maps every point on the hyperbola,

$$\beta_g \beta_h = \frac{p^2}{16\pi^2}, \tag{53}$$

onto itself. The region $\beta_g \beta_h < 1/16\pi^2$ is mapped onto the region $\beta_g \beta_h > 1/16\pi^2$ and vice versa. However, this self-duality property is trivial because the system on which it acts is a product of two decoupled systems. Nonetheless, we mention this here because, as we will show later, at certain values of θ the self-duality will still persist. When $\theta \neq 0$, the two systems are coupled in a nontrivial way. There is a cross coupling between the spin-wave excitaions of one system and the topological excitations of the other system. For instance, the monopole $\tilde{m}(\star x)$ couples to the spin-wave field $m(x)$ and similarly the vortex $m_\mu(x)$ couples to the gauge field ϕ_μ. The main point of this paper is that this coupled system defines an interesting model that has some exact duality symmetries. It also has a rich phase structure as a function of θ. The analysis previously described for the monopole and vortex models can be repeated in the same way as before. The only point to note is that the monopoles and vortices are defined on the dual lattice whereas the gauge fields and the spin variables are defined on the original lattice. However, we can approximately take the point on the dual lattice to coincide with the point on the original lattice. There will be corrections to this approximation but these will involve higher derivative terms which can only affect the short wavelength behavior of the system. The θ term changes the constraint equation for the two auxiliary fields. The new constraint equations become

$$\partial_\mu K_{\mu\nu}=pm_\nu+\frac{p\theta}{2\pi}\tilde{m}_\mu(\star x),$$

$$\partial_\mu K_\mu(x)=pm(x)+\frac{p\theta}{2\pi}\tilde{m}(\star x). \quad (54)$$

The only change is in the inhomogeneous part of these equations,[1]

$$\partial_\mu K^\star_{\mu\nu}=pm_\nu+\frac{p\theta}{2\pi}\tilde{m}_\mu(\star x),$$

$$\partial_\mu K^\star_\mu(x)=pm(x)+\frac{p\theta}{2\pi}\tilde{m}(\star x). \quad (55)$$

Repeating the steps performed for the monopole or the vortex model leads to a θ dependent partition function given by

$$Z_\theta=\mathrm{tr}\exp\left[-\frac{p^2}{8\beta_g}\sum_{xx'}\left(m_\mu(x)+\frac{\theta}{2\pi}\tilde{m}_\mu(x)\right)G(x-x')\left(m_\mu(x')+\frac{\theta}{2\pi}\tilde{m}_\mu(x')\right)\right]\exp\left[-\frac{p^2}{2\beta_h}\sum_{xx'}\left(m(x)+\frac{\theta}{2\pi}\tilde{m}(x)\right)\right.$$
$$\left.\times G(x-x')\left(m(x')+\frac{\theta}{2\pi}\tilde{m}(x')\right)\right]\exp\left(-8\pi^2\beta_g\sum_{xx'}\tilde{m}(x)G(x-x')\tilde{m}(x')\right)\exp\left(-2\pi^2\beta_h\sum_{xx'}\tilde{m}_\mu(x)G(x-x')\tilde{m}_\mu(x')\right)$$
$$\times\exp\left(2\pi i\sum_{x,x'}\tilde{m}(x)G(x-x')\epsilon_{\mu\nu\lambda}\partial_\lambda K^\star_{\mu\nu}\right)\exp\left(-2\pi i\sum_{x,x'}\tilde{m}_\lambda(x)G(x-x')\epsilon_{\lambda\mu\nu}\partial_\mu K^\star_\nu\right). \quad (56)$$

It can be seen that the θ term couples the gauge and spin models and introduces additional interactions in each of them. The partition function of the model is no longer separable into a spin part and a gauge part and cannot be written as

$$Z_\theta\neq Z_1 Z_2. \quad (57)$$

Hence, the phase structure of this model can be quite complicated. One of the immediate consequences of the θ term is that the term representing the Coulomb interaction between electric loops gets modified. This means that the vortex loops [for which $\tilde{m}_\mu(\star x)\neq 0$] acquire an electric charge which is given by

$$Q_\mu(x)=m_\mu(x)+\frac{\theta}{2\pi}\tilde{m}_\mu(\star x). \quad (58)$$

For instance, a loop having only vorticity [$m_\mu(x)=0$, $\tilde{m}_\mu(\star x)\neq 0$] will still have an electric charge given by $(\theta/2\pi)\tilde{m}_\mu(\star x)$. Likewise, the Coulomb interaction between m charges is also modified and acquires a piece due to the monopoles,

$$Q(x)=m(x)+\frac{\theta}{2\pi}\tilde{m}(\star x). \quad (59)$$

Similarly, a point having only non-zero monopole density will have an additional m charge $(\theta/2\pi)\tilde{m}(\star x)$. The electric charges of the vortex loops and the m charges of the monopoles can now take fractional values because of θ. It is convenient to associate a vorticity and an electric charge to every closed loop. The values of these charges are plotted in Fig. 2. The first thing to notice about this model is that the partition function is periodic in θ. This follows from the simple fact that we can always shift the summed variables m_μ and m, as their summation range is infinite. In the presence of a θ term the model is no longer self-dual under the transformations in Eq. (52). However, for certain specific values of θ the model is still self-dual. These are the values for which $2\pi/\theta$ is some integer q. To see the self-duality for these values of θ, we make a simple change of variables:

$$m_\nu+\frac{\tilde{m}_\nu}{q}=-\frac{\tilde{l}_\nu}{q},$$

$$m+\frac{\tilde{m}}{q}=-\frac{\tilde{l}}{q}. \quad (60)$$

Expressing the partition function in terms of $\tilde{l}_\nu$, m_ν, $\tilde{l}$, and

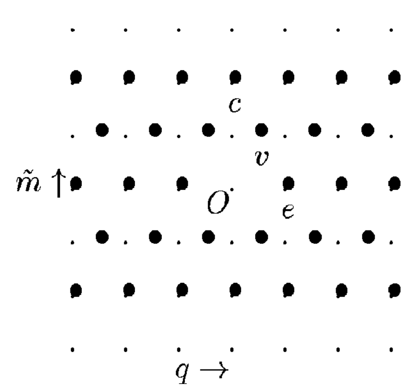

FIG. 2. Charge lattice at $\theta=\pi$.

[1]Since there are many fields involved, our notation might be confusing. See the end of the Appendix for clarification.

m, the partition function reduces to the the original one provided we identify $\tilde{l}_\mu$ with $\tilde{m}_\mu$, $\tilde{l}$ with $\tilde{m}$, and make the following changes:

$$\beta_g \rightarrow \frac{p^2}{16\pi^2 q^2 \beta_h},$$

$$\beta_h \rightarrow \frac{p^2}{16\pi^2 q^2 \beta_g}.$$

Unlike the dual transformations in Eq. (52), the above dual transformations contain more information because they now act on a system which is not a decoupled system. These dual transformations can be used to understand the phase diagram of the model in one region if the phase diagram is known in another region. The points of the phase diagram which are left invariant under the dual transformation are the points on the hyperbola:

$$\beta_g \beta_h = \frac{p^2}{16\pi^2 q^2}. \tag{61}$$

The self-duality in Eq. (61) holds true only when $\theta = 2\pi/q$, q being an integer. We now point out another symmetry that is present in the model for arbitrary values of θ. This symmetry can be deduced by requiring the partition function to be invariant under the transformations

$$m \rightarrow -\tilde{m},$$

$$m_\mu \rightarrow -\tilde{m}_\mu,$$

$$\beta_g \rightarrow \beta_g',$$

$$\beta_h \rightarrow \beta_h',$$

$$\theta \rightarrow \theta'. \tag{62}$$

As the partition function is a trace over the $m, \tilde{m}, m_\mu, \tilde{m}_\mu$ degrees of freedom, it can be written as

$$Z = Z_\theta(m, \tilde{m}, m_\mu, \tilde{m}_\mu). \tag{63}$$

Imposing the condition

$$Z_{\theta'}(-\tilde{m}, -m, \tilde{m}_\mu, m_\nu) = Z_\theta(m, \tilde{m}, m_\mu, \tilde{m}_\mu), \tag{64}$$

we get the following set of equations:

$$8\pi^2 \beta_g + \frac{p^2\theta^2}{8\pi^2\beta_h} = \frac{p^2}{2\beta_h'}, \tag{65}$$

$$\frac{p^2}{2\beta_h} = 8\pi^2 \beta_g' + \frac{p^2\theta'}{8\pi^2\beta_h'},$$

$$2\pi^2\beta_h + \frac{p^2\theta^2}{32\pi^2\beta_g} = \frac{p^2}{8\beta_g'},$$

$$\frac{p^2}{8\beta_g} = \frac{p^2\theta'^2}{32\pi^2\beta_g'} + 2\pi^2\beta_h'.$$

We get four equations in the three variables β_g, β_h, and θ. Interestingly, these four equations are consistent and have the solution

$$\beta_h' = \frac{4\pi^2\beta_h p^2}{64\pi^4\beta_g\beta_h + p^2\theta^2},$$

$$\beta_g' = \frac{4\pi^2\beta_g p^2}{64\pi^4\beta_g\beta_h + p^2\theta^2},$$

$$\theta'^2 = \frac{16\pi^4\theta^2 p^4}{(64\pi^4\beta_g\beta_h + p^2\theta^2)^2}. \tag{66}$$

This means that we can define an exact duality symmetry in the model which now acts on a three coupling space. The duality equations can be recast in a more concise form by defining the variable

$$z = \sqrt{(}\beta_g\beta_h). \tag{67}$$

In terms of the variables z and θ, the duality equations are

$$z'^2 = \frac{16\pi^4 p^4 z^2}{(64\pi^4 z^2 + p^2\theta^2)^2},$$

$$\theta'^2 = \frac{16\pi^4 p^4 \theta^2}{(64\pi^4 z^2 + p^2\theta^2)^2}. \tag{68}$$

If we define the complex coupling ζ by

$$\zeta = \frac{4\pi z}{p} + \frac{i\theta}{2\pi}, \tag{69}$$

the duality transformation can be expressed as

$$\zeta \rightarrow \frac{1}{\zeta}. \tag{70}$$

Since the action is also periodic in θ with period 2π, the transformation

$$\zeta \rightarrow \zeta + i \tag{71}$$

is also a symmetry of the model. These two transformations do not commute with each other and generate the group SL(2,Z). This symmetry group was first pointed out in [8] in the four-dimensional model. In the three-dimensional model that we are considering here, the duality symmetry has a slightly. more complicated form as in Eq. (66) and the group SL(2,Z) acts on the variable z which is given by Eq. (67).

We now proceed to study the phase structure of this model. The phase structure of this model will be a function of the three parameters β_g, β_h, and θ at a given value of p. The various phases of the model will be characterized by the behavior of the $m(x),\tilde{m}(x),m_\mu(x),\tilde{m}_\mu(x)$ excitations. Depending on the density of these excitations, the phases in this model will have different physical properties. In order to arrive at the phase structure of the model, we will use simple free energy arguments based on the energy and the entropy of loops. Though these arguments are admittedly crude and ought to be substantiated by other methods, they do provide us with a qualitative picture of the phase structure. We first note that there will be regions in parameter space where the loops will condense. There are two kinds of loops that can condense. They are labeled by $m_\mu(r)$ and $\tilde{m}_\mu(r)$; $m_\mu(r)$ loops are the world lines of electrically charged particles, and $\tilde{m}_\mu(r)$ loops are the vortex excitations. A loop having both $m_\mu(r)$ and $\tilde{m}_\mu(r)$ is also possible and it will be referred to as a composite loop. These composite loops are formed by the binding of a vortex to an elecrtric current loop. In addition to these loops there are also the pointlike excitations labeled by $m(x)$ and $\tilde{m}(x)$. $\tilde{m}(x)$ are the magnetic monopoles and $m(x)$ will be referred to as spin charges. The reason for this terminology is that the $m(x)$ represent source terms for the spin variables. In addition to these excitations there are also excitations which simultaneously carry magnetic charge and a spin charge. These will be referred to as composite charges. The composite charges are formed by the binding of a magnetic monopole to a spin charge. A composite loop or a composite charge is a combination of an ordinary charge and a dual excitation. If we neglect the long range Coulomb interaction, a crude estimate for the free energy of a loop of length L having charges $(m_\mu,\tilde{m}_\mu)$ is

$$F(L)=\left[\frac{2\pi^2p^2}{\beta_g}\left(m+\frac{\theta}{2\pi}\tilde{m}\right)^2+2\pi^2\beta_h(\tilde{m})^2\right]G(0)L-(\log c)L. \tag{72}$$

Condensation of loops having charges $(m_\mu,\tilde{m}_\mu)$ occurs whenever the loop free energy becomes negative. The constant c depends on the coordination number of the lattice and is approximately 5 for a three-dimensional cubic lattice. In the above approximation, only the self-energy of the loop has been considered in the expression for the free energy. Though this is quite a severe approximation, we expect it to reproduce the general features of the phase diagram. As we have already noted before, the effect of the θ term is to give an electric charge to the vortex lines as

$$q_\nu=m_\nu+\frac{\theta}{2\pi}\tilde{m}_\nu. \tag{73}$$

Similarly, the magnetic monopoles get an effective spin charge

$$q=m+\frac{\theta}{2\pi}\tilde{m}. \tag{74}$$

We first consider the phase diagram at $\theta=0$. When $\theta=0$ the partition function can be decoupled into two partition functions

$$Z=Z_1Z_2, \tag{75}$$

and each of these systems can be considered individually. If we consider Z_1 first, the free energy of a loop of length L carrying electric charge m_μ is given by

$$F(L)=\frac{2\pi^2p^2}{\beta_g}m^2G(0)L-(\log c)L. \tag{76}$$

Thus, $F(L)<0$ if

$$\beta_g>\frac{2\pi^2p^2G(0)}{\log c}. \tag{77}$$

Therefore, the model described by Z_1 will exist in two phases, a small β_g phase in which the current loops are very sparse and a large β_g phase in which the current loops are very dense. The other excitations in this model, the magnetic monopoles, will have a density given by

$$\rho(\beta_g)=\exp[-8\pi^2\beta_gG(0)\tilde{m}^2(x)]. \tag{78}$$

Unlike the density of current loops, the magnetic monopole density falls continuously to zero as the coupling constant β_g is increased. We can make a similar analysis of the model Z_2. The same free energy arguments applied to the vortex loops give $F(L)<0$ if

$$\beta_h<\frac{\log(c)}{2\pi^2G(0)}. \tag{79}$$

The small β_h phase has a high density of vortex loops whereas the large β_h phase has a very low density of vortex loops. The spin charges m change continuously with β_h as

$$\rho(\beta_h)=\exp\left(-\frac{p^2G(0)m^2}{2\beta_h}\right). \tag{80}$$

For the coupled model, this analysis can be repeated but there are now three coupling constants β_g, β_h and θ. We consider the condensation condition given by Eq. (72) for various limiting values of β_g and β_h. If we fix $\theta=\pi$, the different excitations present in the system are those which correspond to the black points in the charge lattice in Fig. 2. The excitations corresponding to these black points are multiples of the following fundamental excitations: (1) electric current loops (1,0), which have a nonzero electric charge; (2) vortex loops (0,1); which have a nonzero electric charge because of Eq. (73); (3) Composite loops (1,−2); which have exactly zero electric charge, again because of Eq. (73).

A similar charge lattice can be drawn for the pointlike following excitations in the model. The excitations of this charge lattice are multiples of the excitations: (1) spin charges (1,0), which have only an m charge; (2) magnetic monopoles (0,1), which have a nonzero m charge because of Eq. (74). (3) Compositely charged pointlike objects $(1,-2)$, which have a zero m charge because of Eq. (74).

We have to compare the free energies of these condensates and then choose the one with the lowest free energy. First, we can explore various limits of the coupling space.

(1) $\beta_h=0$. The free energies of the possible condensates are given by

$$F(L)/L=\frac{p^2}{8\beta_g}G(0)\left(m+\frac{1}{2}\tilde{m}\right)^2-\log(c). \tag{81}$$

(a) $(1,0)$:$p^2/8\beta_g G(0)-\log(c)$.
(b) $(0,1)$:$1/32\beta_g G(0)-\log(c)$.
(c) $(1,-2)$:$-\log(c)$.

The above estimates show that the composite loops $(1,-2)$ will always have the lowest free energy. Therefore, in this limit, there is always a condensate of composite loops $(1,-2)$ and there will be no phase transition on this axis.

(2) $\beta_h=\infty$. The general free energy relation Eq. (72) shows that this limit forces $\tilde{m}_\mu\approx 0$. The free energy condition becomes

$$F(L)=\frac{p^2}{8\beta_g}m^2G(0)-\log(c). \tag{82}$$

Since $\tilde{m}$ is forced to be zero, we only have to consider electric (1,0) loops. $F(L)$ of (1,0) loops becomes negative for $\beta_g>G(0)/8\log(c)$. This implies that there is a phase transition on this axis from a small β_g phase containing very few current loops to a large β_g phase containing very large current loops.

(3) $\beta_g=0$. In this limit we get the following constraint:

$$\left(m+\frac{1}{2}\tilde{m}\right)=0. \tag{83}$$

This leaves only the composite loops $(1,-2)$ as possible excitations. The free energy condition becomes

$$F(L)=8\pi^2\beta_h G(0)-\log(c). \tag{84}$$

Thus, $(1,-2)$ loops will condense for

$$\beta_h<\frac{\log(c)}{8\pi^2G(0)}. \tag{85}$$

Therefore, on this axis we expect a small β_h phase in which $(1,-2)$ loops condense and a large β_h phase in which the density of $(1,-2)$ loops is very small.

(4) $\beta_g=\infty$. The free energy condition is

$$F(L)=2\pi^2\beta_h\tilde{m}^2G(0)-\log(c). \tag{86}$$

(a) $(1,0)$:$F(L)=-\log(c)$.
(b) $(0,1)$:$F(L)=2\pi^2\beta_h G(0)-\log(c)$.
(c) $(1,-2)$:$F(L)=8\pi^2\beta_h G(0)-\log(c)$.

It is clear that the condensate (1,0) always has the lowest free energy. Therefore, the electric current loops are always dense on this axis and there is no phase transition in this limit. The other special case to be considered is the following.

(5) $\beta_g=\beta_h$. In this limit the free energy condition becomes

$$F(L)=\frac{p^2}{8\beta}\left(m+\frac{1}{2}\tilde{m}\right)^2G(0)+2\pi^2\beta\tilde{m}^2G(0)-\log(c). \tag{87}$$

The condensation criterion can be written as the interior of the ellipse on the charge lattice

$$\left(\frac{q^2}{a^2}+\frac{\tilde{m}^2}{b^2}\right)<1. \tag{88}$$

The major and minor axes of the ellipse are given by

$$a^2=\frac{8\beta\log(c)}{G(0)p^2}, \tag{89}$$

$$b^2=\frac{\log(c)}{2\pi^2\beta G(0)}.$$

When $a\gg 1$ and $b\ll 1$, the ellipse is very flat along the q axis and the condensate with the lowest free energy is (1,0). When $a\ll 1$ and $b\gg 1$, the ellipse is very flat along the $\tilde{m}$ axis and the condensate with the lowest free energy is $(1,-2)$. For values of a and b such that they are comparable, the free energy is the lowest for the (0,1) condensate. Therefore, we generically expect three phase transitions on this axis. This case occurs in [7] in the four-dimensional model. It is interesting to note that exactly the same condensation condition appears in the three-dimensional model, the only difference being that the four-dimensional Green's function gets replaced by the three-dimensional one and the entropy factor takes a different value.

Now that we have discussed the condensates in the various regions of parameter space, we can propose the phase diagram of this model at $\theta=\pi$. It is shown in Fig. 3. So far we have only discussed phase transitions of the electric, vortex, and composite loops. We have already noted that there are other excitations present in this model. These are the point like excitations, magnetic monopoles, m charges, and the composite charges of m and $\tilde{m}$. The density of these excitations changes continuously with β_g, β_h, and θ. The density of the magnetic monopoles and the m charges is given by

$$\rho_{m,\tilde{m}}=\exp[-8\pi^2\beta_g G(0)\tilde{m}^2]\exp\left[\frac{-p^2}{2\beta_h}G(0)\left(m+\frac{1}{2}\tilde{m}\right)^2\right]. \tag{90}$$

This is one of the special features of the model in three dimensions. As these excitations are point like, they form a three-dimensional Coulomb gas of charged objects whose

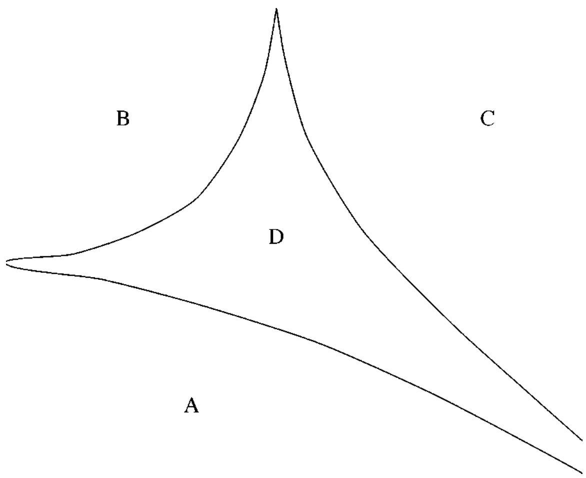

FIG. 3. Schematic phase diagram at $\theta=\pi$. The X axis is the β_g coupling and the Y axis is the β_h coupling.

density is always nonzero. The densities of the three possible types of pointlike excitations in this model are given by the following.

(1) (0,1): magnetic monopoles

$$\rho_{\tilde{m}} \approx \exp -\left(8\pi^2\beta_g + \frac{p^2}{8\beta_h}\right) G(0). \tag{91}$$

(2) (1,0): spin charges

$$\rho_m \approx \exp\left(-\frac{p^2}{2\beta_h} G(0)\right). \tag{92}$$

(3) $(1,-2)$: composite charges

$$\rho_c \approx \exp[-64\pi^2\beta_g G(0)]. \tag{93}$$

Since each of these densities is always nonzero, we have a three-dimensional Coulomb gas containing three species of charged particles. The densities of these three species of charged particles determine various correlation functions (which are defined later on) independently, and can lead to different effects at different values in the parameter space. The behavior of physical correlation functions will be determined by the density of electric and vortex loops as well as the density of magnetic monopoles and m charges. The various regions in the schematic phase diagram in Fig. 3 are discussed below.

(A) In this region only the composite loops $(1,-2)$ condense. At large values of β_h, the density of magnetic monopoles and spin charges is large small whereas at small values of β_h they become nonzero but obey the relation

$$m + \frac{\tilde{m}}{2} = 0. \tag{94}$$

The density of magnetic monopoles decreases as β_g is increased.

(B) In this region the electric current loops (1,0), the vortex loops (1,0), and the composite loops $(1,-2)$ have almost zero density. Therefore, this phase is essentially free of all loop excitations. The magnetic monopoles (0,1) and spin charges (1,0) have a large density at the top left corner of the phase diagram which decreases as as we go away from it in any direction.

(C) In this region the electric loops (1,0) condense whereas vortex loops (0,1) have very low density. The magnetic monopoles (0,1) have a very low density and the spin charges (1,0) have a large density at the top tight hand corner of the phase diagram.

(D) In this region the vortex loops (0,1) condense and the electric current loops (1,0) have a very low density. The magnetic monopoles (0,1) and spin charges (1,0) both have a density which is not very large or very small.

We will now show that these different phases can be characterized by the behavior of correlation functions which are of the order-disorder type. Before we do this, we briefly describe how these correlation functions are defined in each of the models discussed in the previous two sections. The correlation functions are simple generalizations of the Wilson loop [13]. In the model described in Eq. (18) we can introduce an external current $J_\mu(x)$ in a loop C on the lattice and a monopole-antimonopole pair at points $\star x_1$ and $\star x_2$. The monopole pair is introduced by choosing $t_{\mu\nu}=1$ on a string joining $\star x_1$ and $\star x_2$. The correlation function is defined by

$$W(C,x_1,x_2) = \frac{Z_1(m_\mu \to m_\mu + J_\mu, s_{\mu\nu} \to s_{\mu\nu} + s'_{\mu\nu}(x))}{Z_1}. \tag{95}$$

This is equivalent to making the following change in $\tilde{m}(\star x)$:

$$\tilde{m}(\star x) \to \tilde{m}(\star x) + \rho(\star x), \tag{96}$$

where ρ is defined as

$$\rho(\star x) = \delta(\star x - \star x_1) - \delta(\star x - \star x_2). \tag{97}$$

The correlation function W measures the free energy of an external current loop J_μ on C and an external monopole-antimonopole pair at points $\star x_1$ and $\star x_2$. We can similarly define a correlation function for the model described in Eq. (34) as

$$\tilde{W}(\star C, x_1, x_2) = \frac{Z_2(m \to m + M(x), l_\mu \to l_\mu(x) + l'_\mu(x))}{Z_2}. \tag{98}$$

Here, $l'_\mu \neq 0$ on all the links which are dual to the plaquettes in a surface bounded by the loop $\star C$ on the dual lattice, and $M(x)$ has the form

$$M(x) = \delta(x - x_1) - \delta(x - x_2). \tag{99}$$

The correlation function $\tilde{W}$ measures the free energy of an external vortex loop $\star C$ and an external pair of spin charges at x_1 and x_2. The correlation functions W and $\tilde{W}$ are examples of order disorder variables as they involve both the gauge (spin) degrees of freedom and the monopole (vortex) degrees of freedom. In the interacting model we can consider the following correlation function:

$$W_{int}(C,\star C,x_1,x_2,\star x_1,\star x_2)=\frac{Z(m_\mu\to m_\mu+J_\mu,s_{\mu\nu}\to s_{\mu\nu}+s'_{\mu\nu},m\to m+M,l_\mu\to l_\mu+l'_\mu)}{Z}. \tag{100}$$

In the presence of interactions (when $\theta\neq 0$),

$$W_{in}\neq W\widetilde{W}, \tag{101}$$

where we have suppressed the arguments of the correlation function. This correlation function obeys the same duality invariance as the partition function,

$$W_{int}(C,\star C,x_1,x_2,\star x_1,\star x_2)_{\beta_g,\beta_h,\theta} = W_{int}(\star C,C,\star x_1,\star x_2,x_1,x_2)_{\beta'_g,\beta'_h,\theta'}, \tag{102}$$

where β'_g, β'_h, and θ' are related to the unprimed values by the duality equations in Eq. (66).

It is well known that a dilute gas of magnetic monopoles in three dimensions results in an area law behavior for the Wilson loop [3]. This effect arises from the form of the monopole-current loop interaction. Also, an external monopole-antimonopole pair will experience a screened Coulomb potential in this gas. Since the form of the spin-charge–vortex-loop interaction is the same as the monopole-current loop interaction, a dilute gas of spin charges will result in a similar area law behavior for an external vortex loop. Again, an external pair of m charges will experience a screened Coulomb interaction in this gas. Similarly, composite charges $(1,-2)$ will result in an area law for an external composite Wilson loop that consists of a loop of electric charge (1) and vorticity (-2). The details of this calculation are exactly analogous to those in [3]. However, our model also has looplike excitations which can screen external Wilson loops and vorticity loops which can screen external vortex loops. For instance, a condensate of (1,0) loops can screen a Wilson loop of charge (1,0) and yield a perimeter law. When both current loops and monopoles are present, the Wilson loop will have an area law piece but the perimeter law piece will dominate at large distances. Similarly, when (0,1) vortex loops condense, an external vortex loop of charge (0,1) can be screened by these loops and again result in a perimeter law for the external vortex loop inspite of the the presence of m charges which alone would have yielded an area law. The same holds true for composite $(1,-2)$ loops and $(1,-2)$ charges. From the partition function in Eq. (56), we see that there is no monopole-vortex or spin-charge–current-loop interaction. Thus these excitations cannot influence each other in any drastic way.

V. CONCLUSIONS

In this paper we showed that it is possible to define an analogue of the θ term in three dimensions. We presented our analysis of the θ term in a lattice model which was motivated by the work of the authors in [7,8]. The θ term couples magnetic monopoles to the scalar field and vortices to the gauge field. The phase structure of the model changes as a function of θ. In fact, the interactions in this model arise entirely because of the nonzero value of θ. This model has an exact duality symmetry which acts on a three-parameter space. This seems to be the first example of a statistical model in which the duality transformation acts on a three-coupling space. We also made a qualitative analysis of the phase diagram using energy entropy arguments and we showed that at nonzero values of θ there are phases in which excitations having composite loops condense. A special feature of the three-dimensional model is that there are novel pointlike excitations of different species which form a three-dimensional Coulomb gas. Our analysis is also instructive in understanding how the dual transformation works in systems containing both vortices and monopoles. Since the phase diagram of this model (which was studied at $\theta=\pi$) admits phases with charged vortices, it will be interesting if some condensed matter systems can be described by this model. Another interesting avenue is to introduce θ-like terms in non-Abelian gauge theories (in three dimensions) and look for oblique phases at nonzero values of θ.

APPENDIX

In this appendix we will examine the form of the monopole–current-loop and the vortex–m-charge interaction. The monopole–current-loop interaction is given by

$$\sum_{x,x'}\widetilde{m}(\star x)G(x-x')\epsilon_{\mu\nu\lambda}\partial_\mu K_{\nu\lambda}(x'). \tag{A1}$$

$K^\star_{\mu\nu}$ is a particular solution of

$$\partial_\mu K^\star_{\mu\nu}=pm_\nu. \tag{A2}$$

Consider a configuration in which the monopole is along the z axis and at a distance z and there is a circular current loop (of radius R) is in the X-Y plane, i.e., only $m_x,m_y\neq 0$. It is easily seen that a particular solution of Eq. (A2) is

$$K_{xy}=-p \quad \text{inside } C,$$

$$K_{xy}=0 \quad \text{otherwise},$$

$$K_{xz}=K_{yz}=0. \tag{A3}$$

This reduces the monopole–current-loop interaction to

$$-2\sum_{x,x'}\widetilde{m}(\star x)\partial_{z'}G(x-x')K^\star_{xy}(x'). \tag{A4}$$

The coordinates of x and x' are given by

$$x(0,0,z),$$

$$x'(x',y',0). \tag{A5}$$

Using

$$\partial_{i'} G(x-x') = \frac{(x-x')_i}{|x-x'|^3}, \tag{A6}$$

the interaction is given by the expression

$$2pm \int_s \frac{z dx' dy'}{(x'^2+y'^2+z'^2)^{3/2}}. \tag{A7}$$

The integration is over the area of the loop. The evaluation of this integral is straightforward and gives

$$I = 2pm\{2\pi[1-\cos(\alpha)]\}, \tag{A8}$$

where α is the azimuthal subtended by the current loop at the monopole.

Now we turn to the vortex-m interaction. It has the form

$$-\sum_{x,x'} \tilde{m}_\lambda(x) G(x-x') \epsilon_{\lambda\mu\nu} \partial_\mu K^\star_\nu(x'). \tag{A9}$$

$K^\star_\mu$ is the solution of the inhomogeneous equation

$$\partial_\mu K^\star_\mu(x) = pm(x). \tag{A10}$$

Consider a configuration in which $m(x) \neq 0$ at a point on the z axis a distance L from the origin. The solution of the inhomogeneous equation is chosen to be

$$K^\star_x = K^\star_y = 0,$$

$$K^\star_z = p\Theta(z-L). \tag{A11}$$

The Θ function has the property

$$\Theta(z) = 1, \quad z > 0$$

$$\Theta(z) = 0, \quad z < 0. \tag{A12}$$

The coordinates of the points x and x' are given by

$$x(R\cos(\theta), R\sin(\theta), 0),$$

$$x'(0,0,z'). \tag{A13}$$

Again using Eq. (A6) we are led to the integral

$$-2\pi mpR^2 \int_L^\infty \frac{dz'}{(R^2+z'^2)^{3/2}}. \tag{A14}$$

This integral is also straightforward to evaluate and gives the solid angle interaction

$$I = -2pm\{2\pi[1-\cos(\alpha)]\}, \tag{A15}$$

which is just the negative of the monopole–current-loop interaction.

The purpose of this appendix was to show that the monopole–current-loop interaction has the same form as the interaction between the m charge and the vortex loop apart from a sign factor.

A note on notation. In our model there are several fields: $\phi_\mu(x), \theta(x)$ are the gauge and spin variables; $m_\mu(x), m(x)$ are the charged fields; $\tilde{m}(\star x), \tilde{m}_\mu(\star x)$ are the topological excitations (with the tildes); $\phi(x), \tilde{\phi}_\mu(x)$ [which appear in the solutions of the constraint, Eqs. (25) and (39)] make their appearance at intermediate stages and we hope that these are not confused with the other fields. Also, θ as a coupling constant should be distinguished from the spin variable $\theta(x)$.

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